FREE VIBRATION OF TWO-DIMENSIONAL FUNCTIONALLY GRADED SANDWICH NANO BEAMS

DAO ĐỘNG TỰ DO CỦA DẦM SANDWICH NANO CÓ CƠ TÍNH BIẾN THIÊN HAI CHIỀU

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Abstract: *In this paper, the free vibration of two-dimensional functionally graded (2D-FG) sandwich nanobeam is investigated by the finite element method. The material properties of 2D-FG sandwich nanobeam are assumed to vary in both axial and thickness directions according to a power law. Based on Eringen's nonlocal elasticity theory, the governing equations of motion are derived. A parametric study has been carried out to show the effect of material distribution, nonlocal effect, on the natural frequencies of the beams. The finite element method is employed to establish the equations and compute the vibration characteristics of the beam.*

Keywords: *two-dimensional functionally graded, nanobeams, nonlocal model, free vibration, finite element method, sandwich beams.*

Tóm tắt: *Trong bài báo này, dao động tự do của dầm nano sandwich có cơ tính biến thiên hai chiều được nghiên cứu bằng phương pháp phần tử hữu hạn. Tính chất vật liệu của dầm biến đổi theo chiều dọc và chiều cao của dầm bằng quy luật số mũ. Phương trình chuyển động của dầm được thiết lập dựa trên lý thuyết đàn hồi không địa phương. Ảnh hưởng của các tham số vật liệu, tham số không địa phương, tỉ số của các lớp sandwich của dầm đến tần số của dầm cũng được chỉ ra. Phương trình phần tử hữu hạn được thiết lập để tính toán đặc trưng dao động của dầm.*

Từ khóa*: cơ tính biến thiên hai chiều, dầm nano, mô hình không địa phương, dao động tự do, phương pháp phần tử hữu hạn, dầm sandwich.*

1. Introduction

Sandwich beams are widely used in the aerospace industry as well as in other industries due to their high stiffness to weight ratio. Functionally graded materials (FGMs), initiated by Japanese scientists in 1984, are employed to fabricate functionally graded sandwich (FGSW) beams to improve their performance in severe conditions. Investigations on the mechanical behavior of the FGSW beams have been recently reported by several researchers [12,13,14].

The nonlocal field theory, one of the talented continuum models in nanomechanics considering size-dependent effects, was first developed by Eringen [1,2,3]. Most classical continuum theories are based on hyperelastic constitutive relations which assume that the stress at a point is functions of strain at the point. On the other hand, the nonlocal continuum mechanics assumes that the stress at a point is a function of strains at all points in the continuum. By using this theory, the equilibrium differential and motion equations for nanostructures can be derived. Several investigations on vibration of nanobeams have been reported in the literature. Reddy [4] studied the bending, buckling and vibration of homogenous nanobeams. A compact generalized beam theory is used by Aydogdu [5] to analyze the bending, buckling and vibration of nanobeams. Simsek and Yurtcu [6] derived analytical solutions for bending and buckling of FG nanobeams. Based on the finite element method, Eltaher et al. [7,8] studied free vibration of FG size-dependent nanobeams. On the basis of the nonlocal differential constitutive relations of Eringen, Zemri, Houari, Bousahla and Tounsi [9] proposed a nonlocal shear deformation beam theory to study the bending, buckling, and vibration of FG nanobeams. Rahmani and Pedram [10] simultaneously studied bending and buckling of Timoshenko FGM nanobeams using analytical methods.

In this paper, the free vibration of two-dimensional functionally graded (2D-FG) sandwich nanobeams is studied. Timoshenko beam theory incorporated with nonlocal differential equation of

Eringen is used to derive the nonlocal differential equations of motion and the finite element method is employed to compute the frequency parameters of the beam. The effect of nonlocal parameter, FG power indexes and layer thickness ratio on the vibration characteristics is examined and discussed.

2. Problem and formulation

2.1 Grade functionally sandwich nanobeam

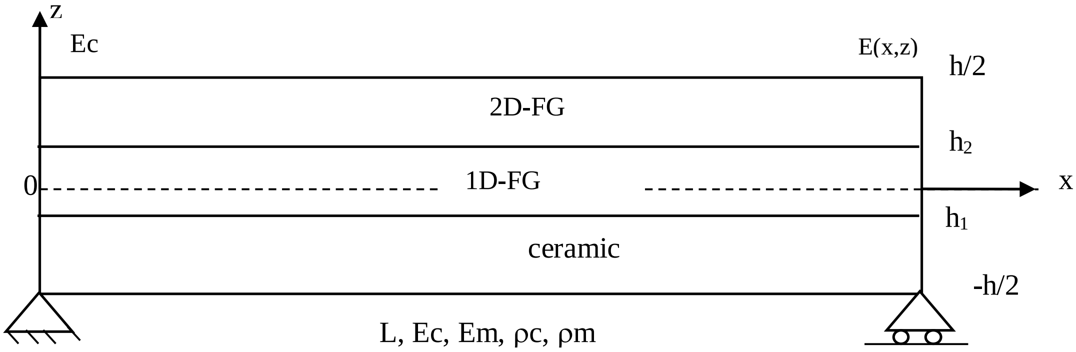

Figure 1. *The model of the bi-dimensional FG sandwich nanobeam*

Figure 1 shows a bi-dimensional FG sandwich nanobeam with a length of *L*. Assuming the sandwich layers are adhesion-solid and non-slip. The upper face of the sandwich beam is made of 2D-FG, the lower face is made of ceramic and the core is made of 1D-FG as shown in Fig. 1. In the Figure, a Cartesian coordinate system (*x, z*) is introduced such that the *x*-axis is on the mid-plane, and the *z*-axis is perpendicular to the mid-plane, and it is directed upward. The beam cross-section is assumed to be rectangular with width *b* and height *h*. The beam material is assumed to be formed from ceramic and metal whose volume fraction varies in both the thickness and longitudinal directions, the power law of volume fraction of the ceramic is given as [11]:

$$\begin{cases} V_c^3(x,z) = 1 & z \in \left[-\dfrac{h}{2}; h_1\right] \\ V_c^2(x,z) = \left(\dfrac{z-h_2}{h_1-h_2}\right)^{nz} & z \in [h_1; h_2] \\ V_c^1(x,z) = \left(1-\dfrac{x}{2L}\right)^{nx}\left(\dfrac{2(z-h_2)}{h-2h_2}\right)^{nz} & z \in \left[h_2; \dfrac{h}{2}\right] \end{cases} \tag{1}$$

where *nz* and *nx* are the grading indexes, which dictate the variation of the constituent materials in the thickness and longitudinal directions, respectively. $V_c^k(x,z)$ (k = 1,2,3) is the volume fraction function of ceramic material of the k-th layer. Consider the 2D-FG sandwich nano beam, distributed equally in the ceramic and metal phases, the modified rule of mixture can be written as [11]:

$$P^k(x,z) = P_m V_m^k + P_c V_c^k \tag{2}$$

the $()_c$ and $()_m$ subscripts respectively denote the ceramic and metal. The effective material properties *(P)* (such as Young's modulus and mass density, etc.) for the sandwich nanobeams can be written.

$$P^k(x,z) = P_m + (P_c - P_m)V_c^k(x,z) \tag{3}$$

where *P*c, *P*m denote the properties of ceramic, metal, respectively, P^k denote the effective material properties of the *k-th* layer.

2.2 Government Equations

The axial displacement u and transverse displacement w at any point based on Timoshenko beam theory are given t by.

$$\begin{aligned} u(x,z,t) &= u_0(x,t) + z\theta(x,t), \\ w(x,z,t) &= w_0(x,t) \end{aligned} \tag{4}$$

where u_0 and w_0 are the axial and transverse displacements at the midplane, and *t* is the time, *θ* is the angle of rotation of the cross-section. From there, we get the strain components:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z\frac{\partial \theta}{\partial x}$$
$$\gamma_{xz} = \frac{\partial w_0}{\partial x} + \theta \tag{5}$$

Hamilton's principle has the form.

$$\int_{t_1}^{t_2} \left(\delta U - \delta T\right)dt = 0 \tag{6}$$

where δU and δT are the virtual strain and kinetic energies. They can be described as:

$$\delta U = \int_0^L \left(N\frac{\partial \delta u_0}{\partial x} + M\frac{\partial \delta\theta}{\partial x} + Q\left(\frac{\partial \delta w_0}{\partial x} + \delta\theta\right)\right)dx \tag{7}$$

where *N, M, Q* are the axial normal force, the bending moment and the shear force, respectively:

$$N = b\int_{-h/2}^{h/2} \sigma_{xx}(x,z)\,dz, \quad M = b\int_{-h/2}^{h/2} z\sigma_{xx}(x,z)\,dz, \quad Q = b\int_{-h/2}^{h/2} k_s\sigma_{xz}(x,z)\,dz \tag{8}$$

$$\delta T = \int_0^L \left[I_{11}\left(\frac{\partial u_0}{\partial t}\frac{\partial \delta u_0}{\partial t} + \frac{\partial w_0}{\partial t}\frac{\partial \delta w_0}{\partial t}\right) + I_{12}\left(\frac{\partial u_0}{\partial t}\frac{\partial \delta\theta}{\partial t} + \frac{\partial \delta u_0}{\partial t}\frac{\partial \theta}{\partial t}\right) + I_{22}\frac{\partial \theta}{\partial t}\frac{\partial \delta\theta}{\partial t}\right]dx \tag{9}$$

where $I_{11},\ I_{12},\ I_{22}$ are the mass moments, defined as:

$$\left(I_{11},\, I_{12},\, I_{22}\right) = b\int_{-h/2}^{h/2} \rho^k(x,z)\left(1,\ z,\ z^2\right)dz \tag{10}$$

where, $\rho^k(x,z)$ is the mass density of the *k*-th layer. Substituting Eqs. (7), (9) into Eq. (6), we obtained the following equations of motion:

$$\frac{\partial N}{\partial x} = I_{11}\frac{\partial^2 u_0}{\partial t^2} + I_{12}\frac{\partial^2 \theta}{\partial t^2}; \quad \frac{\partial Q}{\partial x} = I_{11}\frac{\partial^2 w_0}{\partial t^2}; \quad \frac{\partial^2 M}{\partial x^2} = I_{12}\frac{\partial^2 u_0}{\partial t^2} + I_{22}\frac{\partial^2 \theta}{\partial t^2} \tag{11}$$

The nonlocal constitute equations for nanobeams can be written in form [1,2]:

$$\sigma_{xx} - \mu\frac{\partial^2 \sigma_{xx}}{\partial x^2} = E^k(x,z)\varepsilon_{xx}$$
$$\sigma_{xz} - \mu\frac{\partial^2 \sigma_{xz}}{\partial x^2} = G^k(x,z)\gamma_{xz} \tag{12}$$

Where $E^k(x,z)$, $G^k(x,z)$ are, respectively, the elastic modulus and shear modulus of the *k*-th layer; $\mu = e_0^2 a^2$ is a nonlocal parameter; e_0 is a constant associated with each material; *a* is an internal characteristic length. Taking the derivative Eq. (11) and using Eqs (5), (12), we obtain:

$$N = A_{11}\frac{\partial u_0}{\partial x} + A_{12}\frac{\partial \theta}{\partial x} + \mu\left(I_{11}\frac{\partial^3 u_0}{\partial x\partial t^2} + I_{12}\frac{\partial^3 \theta}{\partial x\partial t^2}\right),$$
$$M = A_{12}\frac{\partial u_0}{\partial x} + A_{22}\frac{\partial \theta}{\partial x} + \mu\left(I_{11}\frac{\partial^2 w_0}{\partial t^2} + I_{12}\frac{\partial^3 u_0}{\partial x\partial t^2} + I_{22}\frac{\partial^3 \theta}{\partial x\partial t^2}\right) \tag{13}$$
$$Q = A_{33}\left(\frac{\partial w_0}{\partial x} + \theta\right) + \mu\left(I_{11}\frac{\partial^3 w_0}{\partial x\partial t^2}\right)$$

where A_{11}, A_{12} and A_{22}, A_{33} are respectively the axial, axial-bending coupling and bending rigidities and shear rigidities. They are defined as:

$$\left(A_{11}, A_{12}, A_{22}\right) = \int_A E^k(x,z)\left(1, z, z^2\right) dA$$
$$A_{33} = \int_A k_s G^k(x,z) dA \qquad (14)$$

Finally, we can obtain the differential equations of motion for the beam in the forms.

$$\int_{t_1}^{t_2}\int_0^L \left[\left(N\frac{\partial \delta u_0}{\partial x} + M\frac{\partial \delta\theta}{\partial x} + Q\left(\frac{\partial \delta u_0}{\partial x} + \delta\theta\right)\right)\right.$$
$$\left. - I_{11}\left(\frac{\partial u_0}{\partial t}\frac{\partial \delta u_0}{\partial t} + \frac{\partial w_0}{\partial t}\frac{\partial \delta w_0}{\partial t}\right) - I_{12}\left(\frac{\partial u_0}{\partial t}\frac{\partial \delta\theta}{\partial t} + \frac{\partial \delta u_0}{\partial t}\frac{\partial \theta}{\partial t}\right) - I_{22}\frac{\partial \theta}{\partial t}\frac{\partial \delta\theta}{\partial t}\right] dxdt = 0 \qquad (15)$$

2.3 Numerical formulation

The finite element method is employed herein to solve the equations of motion. To this end, the beam is assumed being divided into a number of two-node beam elements with a length of *l*. The vector of nodal displacements *(***d***)* for the element considering the transverse shear rotation γ_0 as an independent variable contains six components as:

$$\mathbf{d} = \{u_1,\ w_1,\ \theta_1,\ u_2,\ w_2,\ \theta_2\} \qquad (16)$$

where $u_1, w_1, \theta_1, u_2, w_2, \theta_2$ are the values of u_0, w_0 and θ_0 at the node 1 and at the node 2. In Eq. (16) and hereafter, a superscript '*T*' is used to denote the transpose of a vector or a matrix**.**

$$u_0 = N_u{}^T\mathbf{d},\quad w_0 = N_w{}^T\mathbf{d},\quad \theta = N_\theta{}^T\mathbf{d} \qquad (17)$$

where *Nu, Nw,* N_θ denote the matrices of shape functions for u_0, w_0, θ, respectively. In this present work, using linear polynomials to interpolate the axial, using Kosmatka polynomials to interpolate transverse displacements and angle rotation. Putting Eq. (16) into the variational statement form Eq. (17), performing integration, the following element equation is obtained:

$$\left(\mathbf{M} + \mathbf{M}_{nl}\right)\ddot{\mathbf{D}} + \mathbf{KD} = 0 \qquad (18)$$

where $\mathbf{D}$ is the total vector displacement, $\mathbf{M}$ and $\mathbf{K}$ are the structural mass and stiffness matrices assembled from the element mass and stiffness matrices over the total elements, respectively; and $\mathbf{M}_{nl}$ is the nolocal mass over the total element. When the nonlocal parameter $\mu = 0$, Eq. (18) returns to the free vibration equation for the conventional beams. In addition, Eq. (18) leads to an eigenvalue problem for determining the frequency ω as:

$$\left[\mathbf{K} - \omega^2\left(\mathbf{M} + \mathbf{M}_{nl}\right)\right]\bar{\mathbf{D}} = 0 \qquad (19)$$

with ω is the circular frequency and $\bar{\mathbf{D}}$ is the vibration amplitude. Eq. (19) leads to an eigenvalue problem, and its solution can be obtained.

3. Numerical result

Numerical investigations are carried out in this section to study the effects of the material distribution (or the power-law indexes) on vibration of the bi-dimensional imperfect FG sandwich nanobeams. To this end, a simply supported FG sandwich nanobeams formed from aluminum (Al), alumina (Al_2O_3) with the material properties (aluminum (Al), E_m = 210 GPa, ρ_m = 7800 kg/m^3, v = 0.23, alumina (Al_2O_3), E_c = 390 GPa, ρ_c = 3960 kg/m^3, v = 0.3 and the layer thickness ratio is defined using three number as (1-1-1), (1-2-1), (2-2-1), (2-1-2) for example (1-1-1) means the thickness ratio of bottom, core, and top layers is 1:1:1. The validation of the derived formulation is necessary to confirm before computing the vibration characteristics of the beam. Since there is no data on the vibration of bi-dimensional FG sandwich nanobeams with the power-law variations of the material properties as considered in this present paper, the fundamental frequency of one-dimensional FG sandwich nanobeams obtained in this present work are compared with the data available in the literature. The beam geometry has the following dimensions: L (length) = 10000.10^{-9} (m), b (width) = 1000.10^{-9} (m) and h=L/20 (thickness). The frequency is normalized (λ_i) as [7].

$$\lambda_i = \omega_i L^2\sqrt{\frac{\rho_c A}{E_c I}} \qquad (20)$$

where ω_i is the *i* th natural frequency of the nanobeam and:

$$A = b.h,\quad I = \frac{bh^3}{12}. \qquad (21)$$

Table 1. *Comparison of fundamental frequency parameter for simply supported 1D-FG nanobeams ($nx = 0$, $h_1 = h_2 = -h/2$)*

	n=1		*n=5*		*n=10*	
μ	**Present**	**[7]**	**Present**	**[7]**	**Present**	**[7]**
1	6.7794	6.7631	5.6975	5.7256	5.4248	5.4425
2	6.4939	6.4774	5.4576	5.4837	5.1964	5.2126
3	6.2417	6.2251	5.2456	5.2702	4.9945	5.0096
4	6.0168	6.0001	5.0566	5.0797	4.8146	4.8286
5	5.8146	5.7979	4.8866	4.9086	4.6527	4.6656

The validation of the derived formulation is necessary to confirm before computing the vibration characteristics of the beam. In Table 1, the fundamental frequencies parameter of a simply supported nanobeam with an aspect ratio L/h=20 and nx=0 obtained in the present paper are compared with the results by Eltaher et al [7]. A good agreement can be noted in Table 1, irrespective of the nonlocal parameter.

The grading index nz versus the fundamental frequency parameters of the FG sandwich nanobeams is illustrated in Figure 2 for L/h = 20, 1-1-1, $\mu=0.10^{-12}$, $\mu=0,1.10^{-12}$, $\mu=0,3.10^{-12}$, $\mu=0,5.10^{-12}$. As seen from the figures, for a given value of the nonlocal parameter, the fundamental frequencies decrease when the nonlocal parameter μ increases, regardless nx parameter.

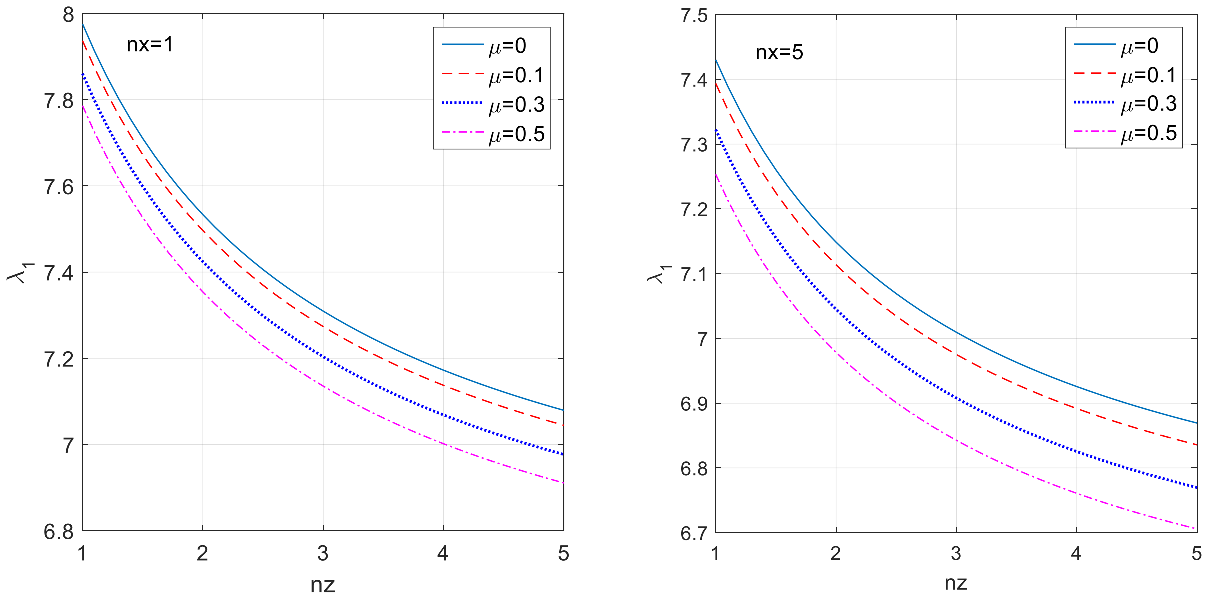

Figure 2. *The variation of the fundamental frequency parameter with material graduation for different nonlocality parameters when L/h = 20, 1-1-1*

Figure 3 shows the non-dimensional frequency of simply supported FG sandwich nanobeam for different values of the FG power indexes when L/h = 20, $\mu=2.10^{-12}$. It is observed that increasing nx and nz decrease the frequency parameter of the sandwich nanobeams which is because increasing the nz and nx power indexes decreases the stiffness of the nanobeams. It is seen that the non-dimensional frequency of FG nanobeams, decreases with the increment of the thickness ratio of core.

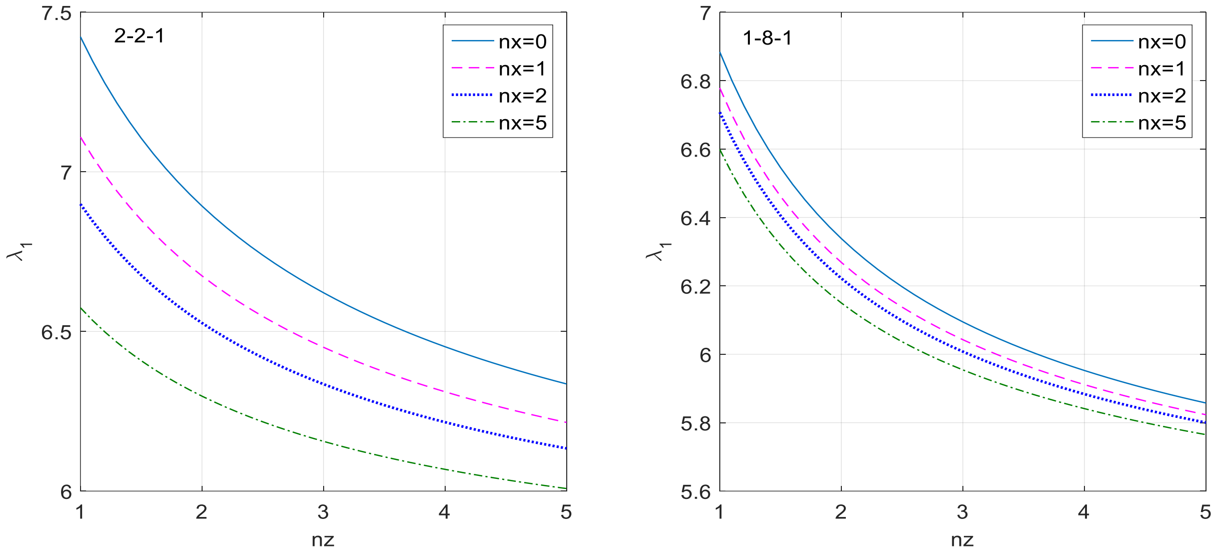

Figure 3. *Non-dimensional natural frequency parameter of 2D-FGM sandwich nanobeams when L/h=20, μ=2.10^{-12}*

The effect of the layer thickness ratio on the frequency parameter can also be seen in Figure 4, it shows the first frequency parameter of sandwich nanobeams for different values of the layer thickness ratio versus the FG *(nz)* power indexes. It is observed that increasing *nz* decreases the frequency parameter of the sandwich nanobeams. It is seen that the first frequency parameter of FG sandwich nanobeams decreases with the increment of the layer thickness ratio value. But the effect of the layer thickness ratio on the first frequency parameter of the FG sandwich nanobeam is complicated.

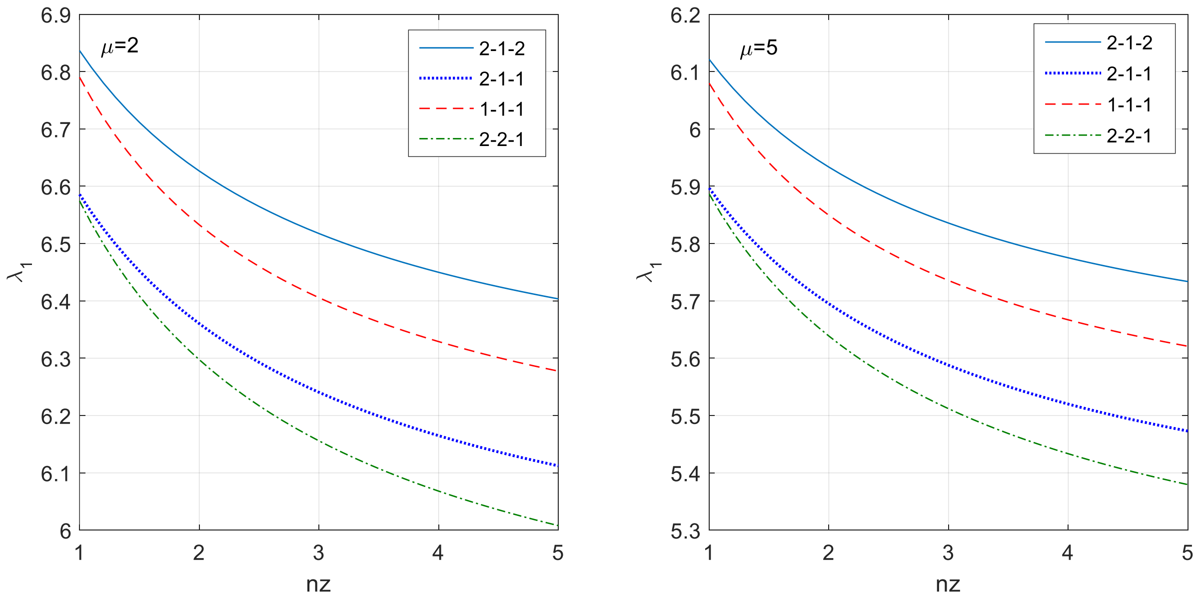

Figure 4. *Aspect nz versus first frequency parameter of 2D-FGM sandwich nanobeams when L/h=20 and nx=5 (μ=2.10^{-12}, μ=5.10^{-12})*

4. Conclusions

The vibration characteristics of 2D-FG sandwich nanobeams are evaluated. The nonlocal Eringen model considering the scale effect is taken into account in the derivation of the equations of motion. The finite element method is employed to discretize the model and to compute the fundamental frequency parameters. A parametric study has been carried out to highlight the effect of nonlocal parameters, FG power indexes and layer thickness ratio on fundamental frequency parameters of the sandwich nanobeam. The obtained numerical results show that, the nonlocal parameter plays an important role in the frequencies of nanobeam, and the frequencies decrease with increasing the nonlocal parameter.

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