DYNAMIC BEHAVIOR OF MULTI-SPAN BI-DIRECTIONAL FUNCTIONALLY GRADED BEAMS UNDER A MOVING LOAD

ĐÁP ỨNG ĐỘNG CỦA DẦM ĐA NHỊP CÓ CƠ TÍNH BIẾN THIÊN HAI CHIỀU DƯỚI TÁC DỤNG CỦA LỰC DI ĐỘNG

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Abstract: *Vibration analysis of multi-span bi-directional functionally graded material (2D-FGM) beams subjected to a moving load is presented by using a high-order deformation theory. The material properties of the beam are assumed to vary continuously in the thickness and longitudinal directions by a power-law distribution. The dynamic response of the beam is computed with the aid of the Newmark method. The obtained numerical result reveals that the variation of the material properties in the thickness and longitudinal directions play an important role on the dynamic response of the beam. A parametric study is carried out to highlight the effect of the material heterogeneity, number of spans on the dynamic response of the beams. The influence of the moving speed is also studied and highlighted.*

Keywords: *2D-FGM, a high-order deformation theory, moving load, finite element method, multi-span.*

Tóm tắt: *Bài báo phân tích dao động của dầm đa nhịp có cơ tính biến thiên hai chiều (2D-FGM) dưới tác dụng của lực di động, bằng lý thuyết dầm bậc cao. Tính chất vật liệu của dầm biến thiên theo chiều dài và chiều dày của dầm bằng quy luật số mũ. Đáp ứng động của dầm được tính toán bằng phương pháp tích phân trực tiếp Newmark. Các kết quả số thu được cho thấy, biến thiên của vật liệu theo hai chiều đóng vai trò quan trọng trong phân tích dao động của dầm. Ngoài ra ảnh hưởng của tham số vật liệu, tham số nhịp dầm, tham số tốc độ của lực di động cũng được nghiên cứu trong bài báo.*

Từ khóa: *2D-FGM, lý thuyết biến dạng bậc cao, tải di động, phương pháp phần tử hữu hạn, dầm đa nhịp.*

1. Introduction

The problems of moving loads on an elastic beam are often meet in the design of bridges, railways, highways and many modern machining operations [1]. A large number of investigations concerning the dynamic analysis of beams subjected to moving load have been reported in the literature; only the main contributions related to the present work are briefly discussed herein. The early and excellent reference is the monograph of Frýba [2], in which several numbers of closed-form solutions for the moving load problems have been derived. Based on the analytical and finite element solutions to a fundamental moving load problem, Olsson [3] provided an interesting discussion and the reference data for studies of the moving load problem. Ichikawa et al. [4] investigated the dynamic behavior of a multi-span continuous beam subjected to a constant speed moving mass.

Functionally graded material, initiated by Japanese scientists in 1984 [5] has received much attention from engineers and researchers. FGM is formed by varying the percentage of constituent materials in certain desired spatial direction. FGM is being used widely as a structural material, and analysis of structures made of FGM is presently an important topic in the field of structural mechanics. A comprehensive list of publications on analyses of FGM structures subjected to different loadings is given in a review paper by Birman and Byrd [6].

There are practical circumstances, in which the unidirectional FGMs may not be so appropriate to resist multi-directional variations of thermal and mechanical loadings. On the other hand, a new type of functionally graded material (FGM) with material properties varying in two or three directions is needed to fulfil the technical requirements such as the temperature and stress distributions in two or three directions for aerospace craft and shuttles where the conventional FGMs (or 1D-FGM) with material properties which vary in one direction are not so efficient. Several models for bi-dimensional

FGM beams and their mechanical behaviour have been considered recently. In this line of works, Simsek [7] considered the material properties vary in both the length and thickness directions, by an exponential function in vibration study of Timoshenko beam. Polynomials were assumed for the displacement field in computing natural frequencies and the dynamic response of the beam. Wang et al. [8] presented an analytical method for free vibration analysis of a 2D-FGM beam. The material properties are also assumed to vary exponentially in the beam thickness and length. The bending of a two-dimensional FGM sandwich (2D-FGSW) beam was investigated by Karamanli [9] using a quasi-3D shear deformation theory and the symmetric smoothed particle hydro-dynamic method. Nguyen et al. [10] proposed a 2D-FGM beam model formed from four different constituent materials with volume fractions to vary by power-law functions in both the thickness and longitudinal directions. Timoshenko beam theory was adopted by the authors in evaluating the dynamic response of the beam to a moving load.

In this paper, a finite element procedure for vibration analysis of multi-span 2D- FGM beams subjected to a moving point load is proposed. The material properties of the beams are assumed to vary continuously in the thickness and longitudinal direction by a power-law distribution. The discrete equations of motion of the beams are solved by using the Newmark method. A parametric study is carried out to highlight the influence of the material heterogeneity, number of spans and loading parameters on the dynamic response of the beam.

2. Problem and formulation

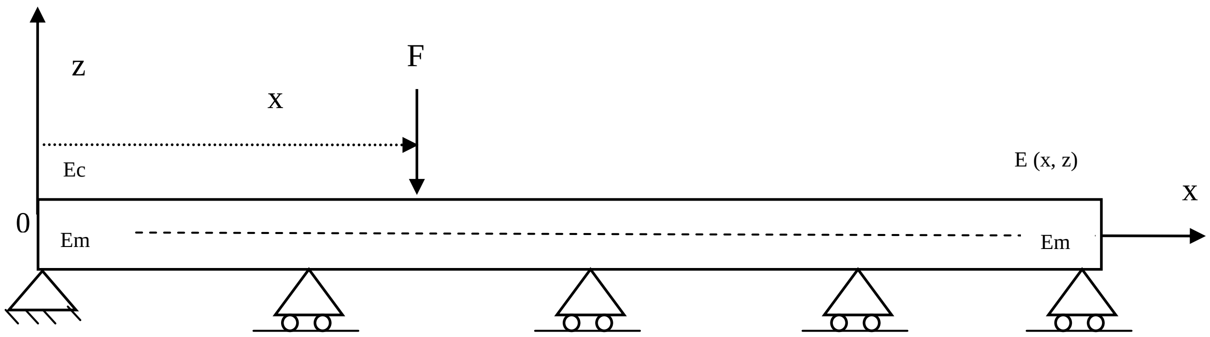

Figure 1. *A multi-span 2D-FGM beam traversed by a moving load*

Figure 1 shows a multi-span beam with length of L subjected to a harmonic load, F= const, moving at a constant speed v from left to right. The beam cross-section is assumed to be rectangular with width b and height h. The beam material is assumed to be a 2D-FGM composed from two constituent materials, and the effective properties of materials are graded in the thickness and longitudinal direction (x, ***z*** -direction) according to a power-law distribution as Karamanli [9]:

$$V_c(x,z)=\left(1-\frac{x}{2L}\right)^{nx}\left(\frac{1}{2}+\frac{z}{h}\right)^{nz};\ \ V_c(x,z)+V_m(x,z)=1;$$

$$P(x,z)=(P_c-P_m)\left(1-\frac{x}{2L}\right)^{nx}\left(\frac{1}{2}+\frac{z}{h}\right)^{nz}+P_m \tag{1}$$

$$-\frac{h}{2}\le z\le\frac{h}{2};\quad 0\le x\le L$$

where V_c and V_m denote the volume fractions of ceramic and metal material, respectively; *P*(x,z) represents the effective material properties, including Young's modulus, shear modulus and mass density; the $()_c$ and $()_m$ subscripts respectively denote the ceramic and metal; *nz* and *nx* are the grading indexes, which dictate the variation of the constituent materials in the thickness and longitudinal directions, respectively.

Based on the third-order shear deformation theory, the axial and transverse displacements at any point of the beam, *u(x,z,t)* and *w(x,z,t),* respectively, are given.

$$u(x,z,t) = u_0(x,t) + z(\gamma_0 - w_{0,x}) - \alpha z^3 \gamma_0$$
$$w(x,z,t) = w_0(x,t). \tag{2}$$

where *t* is the time variable and $\alpha = 4/3\ h^2$, $u_0(x,\ t)$ and $w_0(x,\ t)$ are, respectively, the axial and transverse displacements of the point on the *x*-axis, γ_0 is the transverse shear rotation. The axial strain and shear strain resulted from Eq. (2) are of the forms.

$$\varepsilon_{xx} = u_{0,x} + z\left(\gamma_{0,x} - w_{0,xx}\right) - \alpha z^3 \gamma_{0,x}$$
$$\gamma_{xz} = \gamma_0 - 3\alpha z^2 \gamma_0 \tag{3}$$

Based on the assumption of Hooke's law, the constitutive relation for 2D- FG beam is as follows.

$$\sigma_{xx} = E(x,z).\varepsilon_{xx} = E(x,z)[u_{0,x} + z(\gamma_{0,x} - w_{0,xx}) - \alpha z^3 \gamma_{0,x}]$$
$$\tau_{xz} = G(x,z)\gamma_{xz} = \frac{E(x,z)}{2(1+\upsilon)}\left[\gamma_0 - 3\alpha z^2 \gamma_0\right] \tag{4}$$

where *E(x,z)* and *G(x,z)* are respectively the elastic modulus and shear modulus, which are functions of both the coordinates *x, z,* σ_{xx} and τ_{xz} are the axial stress and shear stress, respectively. The strain energy *U* of the beam resulted from Eqs. (3) and (4) is of the form.

$$U = \frac{1}{2}\int_0^L \begin{bmatrix} A_{11}u_{0,x}^2 + 2A_{12}u_{0,x}(\gamma_{0,x} - w_{0,xx}) + A_{22}(\gamma_{0,x} - w_{0,xx})^2 - 2A_{34}\alpha u_{0,x}\gamma_{0,x} \\ -2\alpha A_{44}\gamma_{0,x}(\gamma_{0,x} - w_{0,xx}) + \alpha^2 A_{66}\gamma_{0,x}^2 + B_{44}\gamma_0^2 \end{bmatrix} dx \tag{5}$$

where A_{11}, A_{12}, A_{22}, A_{34}, A_{44}, A_{66} and B_{44} are the beam rigidities, defined as:

$$(A_{11}, A_{12}, A_{22}, A_{34}, A_{44}, A_{66})(x,z) = \int_A E(x,z)(1, z, z^2, z^3, z^4, z^6)dA$$
$$B_{44}(x,z) = \int_A G(x,z)(1 - 6\alpha z^2 + 9\alpha^2 z^4)dA \tag{6}$$

where *E(x,z), G(x,z)* are respectively the elastic modulus and shear modulus of the beam. The kinetic energy (*T*) of the beam is then given by.

$$T = \frac{1}{2}\int_0^L \begin{bmatrix} I_{11}(\dot{u}_0^2 + \dot{w}_0^2) + I_{22}(\dot{\gamma}_0 - \dot{w}_{0,x})^2 + \alpha^2 I_{66}\dot{\gamma}_0^2 + 2I_{12}\dot{u}_0(\dot{\gamma}_0 - \dot{w}_{0,x}) - 2\alpha I_{34}\dot{u}_0\dot{\gamma}_0 \\ -2\alpha I_{44}\dot{\gamma}_0(\dot{\gamma}_0 - \dot{w}_{o,x}) \end{bmatrix} dx \tag{7}$$

In Eqs. (7), I_{11}, I_{12}, I_{22}, I_{34}, I_{44}, I_{66} are the mass moments, defined as

$$(I_{11}, I_{12}, I_{22}, I_{34}, I_{44}, I_{66})(x,z) = \int_A \rho(x,z)(1, z, z^2, z^3, z^4, z^6)dA \tag{8}$$

where, *ρ(x,z)* is the mass density of the beam. The potential of the moving load is simply given by

$$V_F = -Fw(x,t)\delta(x - vt) \tag{9}$$

in which F, *v* are respectively the amplitude, speed of the moving load and δ (.) is the Kronecker delta.

Using the finite element method, the beam is assumed to be divided into numbers of two-node beam elements of length *l*. The vector of nodal displacements *(***d***)* for the element considering the transverse shear rotation γ_0 as an independent variable contains eight components as.

$$\mathbf{d} = \left\{u_i, w_i, w_{i,x}, \gamma_i, u_j, w_j, w_{j,x}, \gamma_j\right\}^T \tag{10}$$

where $u_i, w_i, w_{i,x}, \gamma_i, u_j, w_j, w_{j,x}, \gamma_j$ are the values of u_0, w_0, $w_{0,x}$ and γ_0 at the node *i* and at the node *j,* respectively. In Eq. (10) and hereafter, a superscript '*T*' is used to denote the transpose of a vector or a matrix**.**

$$\mathrm{u}_0 = \mathrm{N}_u.\mathbf{d};\quad \mathrm{w}_0 = \mathrm{N}_\mathrm{w}.\mathbf{d};\quad \gamma_0 = \mathrm{N}_\gamma.\mathbf{d} \tag{11}$$

with *Nu, Nw* and *Nγ* denote the matrices of shape functions for u_0, w_0 and γ_0, respectively. In the present work, linear shape functions are used for the axial displacement and the shear rotation, using the above interpolation schemes, one can write the strain energy of the beam defined by Eqs. (5) as.

$$U=\frac{1}{2}\sum^{ne}\mathbf{d}^T\mathbf{k}\mathbf{d} \qquad (12)$$

where *ne* is the total number of the elements, and **k** is the element stiffness matrix with the following form.

$$\mathbf{k} = \mathbf{k}_{11}+\mathbf{k}_{12}+\mathbf{k}_{22}+\mathbf{k}_{34}+\mathbf{k}_{44}+\mathbf{k}_{66}+\mathbf{k}_s \qquad (13)$$

with:

$$\mathbf{k}_{11}=\int_0^l N_{u,x}^T A_{11} N_{u,x}dx; \qquad \mathbf{k}_{12}=2\int_0^l N_{u,x}^T A_{12}(N_{\gamma,x}-N_{w,xx})dx;$$

$$\mathbf{k}_{22}=\int_0^l (N_{\gamma,x}-N_{w,xx})^T A_{22}(N_{\gamma,x}-N_{w,xx})dx; \qquad \mathbf{k}_{34}=-2\alpha\int_0^l N_{u,x}^T A_{34} N_{\gamma,x}dx;$$

$$\mathbf{k}_{44}=-2\alpha\int_0^l N_{\gamma,x}^T A_{44}(N_{\gamma,x}-N_{w,xx})dx; \qquad \mathbf{k}_{66}=\alpha^2\int_0^l N_{\gamma,x}^T A_{66} N_{\gamma,x}dx; \qquad (14)$$

$$\mathbf{k}_s=\int_0^l N_\gamma^T B_{44} N_\gamma dx$$

Similarly, the kinetic energy in Eq. (7) can be rewritten as.

$$T=\frac{1}{2}\sum^{ne}\left(\frac{\partial \mathbf{d}}{\partial t}\right)^T \mathbf{m}\left(\frac{\partial \mathbf{d}}{\partial t}\right) \qquad (15)$$

where:

$$\mathbf{m} = \mathbf{m}_{11}+\mathbf{m}_{12}+\mathbf{m}_{22}+\mathbf{m}_{34}+\mathbf{m}_{44}+\mathbf{m}_{66} \qquad (16)$$

is the element consistent mass matrix, in which:

$$\mathbf{m}_{11}=\int_0^l \left(N_u^T+N_w^T\right) I_{11}\left(N_u+N_w\right)dx; \qquad \mathbf{m}_{12}=2\int_0^l N_u^T I_{12}\left(N_\gamma-N_{w,x}\right)dx;$$

$$\mathbf{m}_{22}=\int_0^l \left(N_\gamma-N_{w,x}\right)^T I_{22}\left(N_y-N_{w,x}\right)dx; \qquad \mathbf{m}_{34}=-2\alpha\int_0^l N_u^T I_{34} N_\gamma dx; \qquad (17)$$

$$\mathbf{m}_{44}=-2\alpha\int_0^l N_{\gamma,x}^T\ I_{44}\left(N_y-N_{w,x}\right)dx; \qquad \mathbf{m}_{66}=\alpha^2\int_0^l N_\gamma^T I_{66} N_\gamma dx;$$

Are the element mass matrices stemming from axial, transverse translations, axial translation–sectional rotation coupling, and cross-sectional rotation, respectively. Finally, the potential of the external moving load can be written in the form.

$$V_F=-FN_w^T\delta(x-vt) \qquad (18)$$

Having the element stiffness and mass matrices derived, the equations of motion for the free vibration analysis in the context of finite element analysis can be written in the form.

$$\mathbf{M}\frac{\partial^2\mathbf{D}}{\partial t^2}+\mathbf{KD}=\mathbf{F}^{ex} \qquad (19)$$

Where **D**, **M**, and **K** are the structural nodal displacement vector, mass and stiffness matrices, obtained by assembling the element displacement vector **d**, mass matrix **m**, and stiffness matrix **k** over the total elements, respectively; $\mathbf{F}^{ex}$ is the vector of the nodal external forces.

3. Numerical results and discussion

Using the derived finite element formulation, the dynamic response of multi-span 2D-FG beams is computed in this section. It is assumed that the beam is formed from spans of the same length. Otherwise stated, the beam is assumed to be composed of Steel and Alumina. The Young's modulus, mass density and Poisson's ratio of Steel are respectively 210 GPa, 7800 kg/m3**,** 0.3177, and that of Alumina are 390 MPa**,** 3960 kg/m3 and 0.3, respectively. The beam with *L=20* m**,** *h=1*m and *b***=***0.5* m used by Şimşek and his co-worker [12,13] is chosen in the computations reported below.

3.1 Formulation validation

Validation of the derived formulation is necessary to confirm the accuracy before computing the dynamic response of the beam. Firstly, the natural frequencies of a multi-span homogeneous beam are computed, and the obtained numerical results are listed in Table 1, where the corresponding results obtained by Ichikawa et al [4] are also given. The dimensionless natural frequency parameter, μ_i, in Table 1 is defined as.

$$\mu_i^2 = \omega_i L_s^2 \sqrt{\frac{\rho_0 A}{E_0 I}} \tag{20}$$

Where ω_i are the natural frequencies; L_s is the length of a span; E_0, ρ_0 are Young's modulus and mass density of the homogeneous beam, respectively. It should be noted that since the Bernoulli beam theory is used in Ichikawa et al [4], and in order to enable the numerical results comparable, the frequencies in Table 1 have been computed with an aspect ratio *Ls/h=100*, which is large enough to omit the effect of the shear deformation. As seen from the Table 1, a good agreement between the frequencies computed in the present work with that of Ichikawa et al [4] is noted.

Table 1. *Comparison of first five natural frequencies of multi-span homogeneous beams (nx=0, nz=0)*

Number of spans		μ_1	μ_2	μ_3	μ_4	μ_5
1	Present Ichikawa [4]	3.1414 π	6.2817 2π	9.4202 3π	12.5567 4π	15.6921 5π
2	Present Ichikawa [4]	3.1414 π	3.9261 3.9266	6.2817 2π	7.0661 7.0686	9.4202 3π
3	Present Ichikawa [4]	3.1414 π	3.5560 3.5564	4.2968 4.2975	6.2817 2π	6.7056 6.7076
4	Present Ichikawa [4]	3.1414 π	3.3929 3.3932	3.9261 3.9266	4.4625 4.4633	6.2817 2π

Secondly, the fundamental frequency of a one-span FGM beam composed of Aluminum (Al) and Alumina (Al_2O_3), previously studied in Sina et al [11] and Şimşek [12], is computed. The Young's modulus, mass density and Poisson's ratio of Alumina are *70 GPa*, *2707 kg/m3* and *0.23*, respectively [12]. The computed fundamental frequency parameters of the present work are listed in Table 2 for various values of the aspect ratio, *L/h*. The corresponding values obtained by using an analytical method by Sina et al [11] and a numerical method by Şimşek [12] are also given in the table. The non-dimensional fundamental frequencies, μ, in Table 2 have been defined according to Sina et al [11] as.

$$\mu = \omega L^2 \sqrt{\frac{I_{11}}{h^2 \int_0^L E(z)dz}} \tag{21}$$

Where ω is the fundamental frequency of the FG beam. As seen from the Table 2, the fundamental frequencies computed in the present work are in good agreement with that of Sina et al [11] and Şimşek [12], regardless of the aspect ratios.

Thirdly, the maximum dynamic deflection factor at the mid-span and the corresponding speed of one-span FGM beam composed of Steel and Alumina with L = 20 m, h = 0.9 m and b = 0.4 m, previously studied by Şimşek and Kocatürk [13], are computed. The obtained results are listed in Table 3. In the table, the dynamic deflection factor f_D is defined as $f_D = \max(w(L/2, t)/w_0)$ with w_0 is the static deflection of the steel beam under static load F acting at the mid-span. A very good agreement between the numerical results of the present work with that of Şimşek and Kocatürk [13] is seen from the table.

Table 2. *Comparison of non-dimensional fundamental frequency of one-span FGM beam(nx=0)*

n		**L/h=10**	**L/h=30**	**L/h=100**
0.3	Present	2.7017	2.7382	2.7425
0.3	Sina et al [11]	2.695	2.737	2.742
0.3	Şimşek [12]	2.701	2.738	2.742

Table 3. *Maximum deflection factor and corresponding speed of one-span FGM beam under a moving load (nx=0)*

n	f_D- [13]	f_D-Present	v(m/s)-[13]	v(m/s)- present
0.2	1.0344	1.0377	222	222
0.5	1.1444	1.1476	198	197
1	1.2503	1.2537	179	178
2	1.3376	1.3416	164	163
Pure Alumina	0.9328	0.9379	252	251
Pure Steel	1.7324	1.7418	132	131

The numerical results listed in Tables 1-3 have been computed by using 14 elements for each span. More than 14 elements have been employed, but no improvement in the numerical results have been seen, and in this regard, 14 elements are used to discrete each span in the computations reported below.

3.2 Dynamic deflection

The normalized deflection at the midpoint of the first and second spans of a four-span 2D-FGM beam are shown in Fig. 2 for various values of the index *nx, nz*, speed parameter f_v. In the figures, $w(L_s/2, t)$ denotes the dynamic deflection at the midpoint of the i^{th} span, and $w_0 = FL_{si}^3 / 48E_m I$ is the static deflection of a simply supported beam with length of L_s under a static load F at the midpoint. The speed parameter f_v is defined in accordance with Ichikawa et al [4], $f_v = vL_S\sqrt{\rho_m A / E_m I}$, and thus for the given data of the beam and for f_v = 1.2, the equivalent speed of the moving load is 90 m/s for the beam with a span length of 20 m. As seen from the figures, the material heterogeneity which governed by the index *nx, nz* clearly affects the dynamic deflection of the beam. The maximum normalized deflection of the beam associated with a higher index *nx, nz* is higher than that of the beam with lower index *nx, nz*.

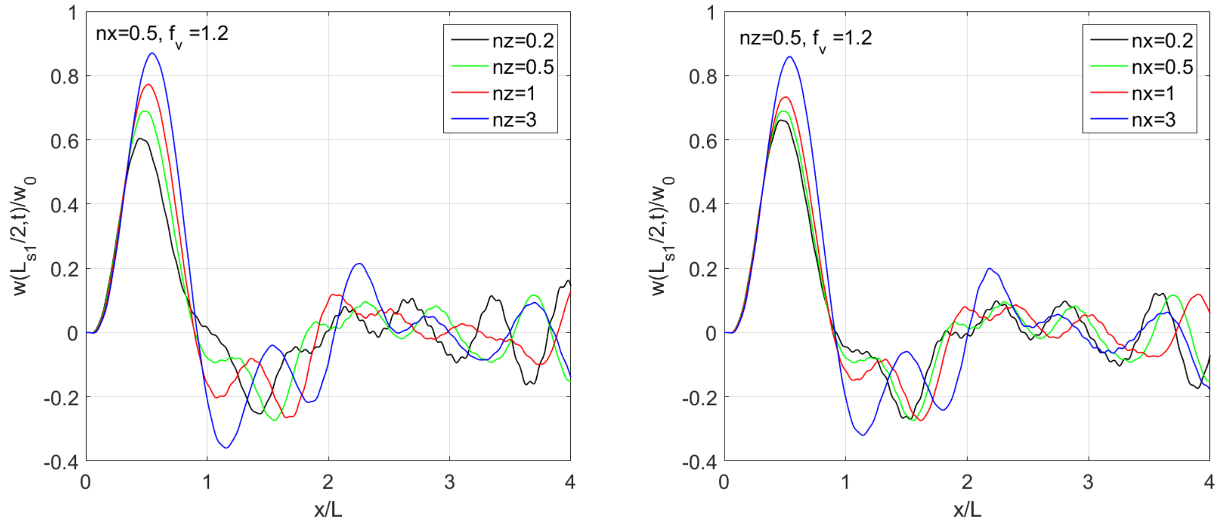

Figure 2. *Normalized deflection at midpoint of the first span (four-span beam)*

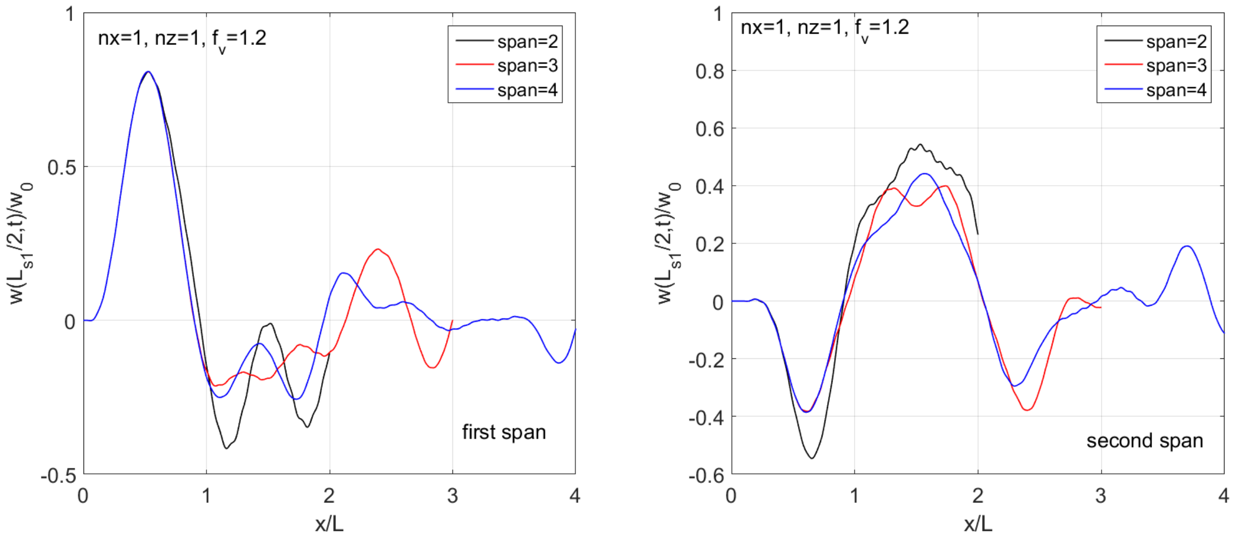

Figure 3. *Normalized deflection at the midpoint of the first and second spans for 2D-FG beam with different numbers of spans (nx=1, nz=1)*

The normalized dynamic deflections at the first and second spans of the 2D-FGM beam with different numbers of spans computed with various values of the speed parameters are shown in Fig. 3 for *nx=1*, *nz=1* and *f_v=1.2*. As seen from the figure, the maximum deflection at the midpoint of the first and the second spans of the beam slightly reduces for the beam with more spans.

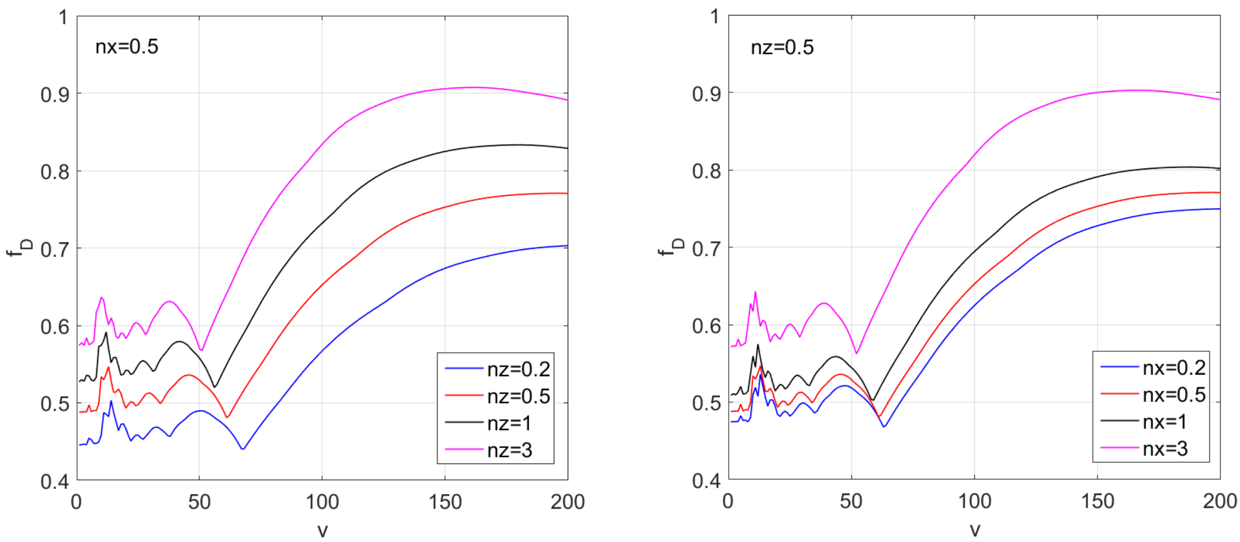

Figure 4. *Maximum normalized deflection at the midpoint of the first span (four-span beam)*

In Fig. 4, the relation between the moving speed v and the maximum dynamic deflection at the midpoint of the first span is shown for various values of the index *nx, nz*. The effect of the material heterogeneity and the moving speed is clearly seen from the figure, and the maximum dynamic deflection is higher for the beam associated with a higher index *nx, nz*, regardless of the moving speed.

4. Conclusion

In this paper, a finite element procedure for vibration analysis of multi-span 2D-FGM beams subjected to a moving point load has been presented. The dynamic responses of the beam have been computed with the aid of the Newmark method. The obtained numerical results have shown that the formulated element is capable to give accurate dynamic characteristics of the beams. A parametric study has been carried out to highlight the influence of material heterogeneity, the number of spans and the loading parameters on the dynamic response of the beam. It has been shown that the beam associated with lower index *nx, nz* endures a smaller dynamic deflection than that of the beam with higher index *nx, nz*.

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